Linear Algebra I

11/11/2022, Friday, 15:00 – 17:00

1 Null space and rank $\qquad$ $12 + 4 = 16$ pts

Consider the matrix $M = \begin{bmatrix} 4 & 3 & 2 & 6 \\ 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

(a) Find the null space of M.

(b) Determine the rank of M.

REQUIRED KNOWLEDGE: **Null space, reduced row echelon form, rank-nullity theorem.**

SOLUTION:

1a: The null space of M is defined as $N(M) = \{\boldsymbol{x} \mid M\boldsymbol{x} = \mathbf{0}\}$. To solve the system of linear equations $M\boldsymbol{x} = \mathbf{0}$, we can put the matrix M into reduced row echelon form:

$$\begin{bmatrix} 4 & 3 & 2 & 6 \\ 2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{(1) \leftrightarrow (3)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 4 & 3 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 4 & 3 & 2 & 6 \end{bmatrix} \xrightarrow[(3) \leftarrow (3) - 4\cdot(1)]{(2) \leftarrow (2) - 2\cdot(1)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -2 & -2 \end{bmatrix} \xrightarrow{(3) \leftarrow (3) + 1\cdot(2)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow{(3) \leftarrow -\frac{1}{3}\cdot(3)} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[(1) \leftarrow (1) - 1\cdot(3)]{(2) \leftarrow (2) + 1\cdot(3)} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{(1) \leftarrow (1) - 1\cdot(2)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, the lead variable are the first three and the fourth is a free variable. As such, we obtain the general solution as

$$\boldsymbol{x} = \begin{bmatrix} -a \\ 0 \\ -a \\ a \end{bmatrix}.$$

This results in

$$N(M) = \left\{ \begin{bmatrix} -a \\ 0 \\ -a \\ a \end{bmatrix} \mid a \in \mathbb{F} \right\}.$$

1b: From the rank-nullity theorem, we know that

$$\operatorname{rank} M + \operatorname{null} M = 4.$$

Since $\operatorname{null} M = 1$, we obtain $\operatorname{rank} M = 3$.

2 Eigenvalues and eigenvectors

$4 + 8 + 12 = 24$ pts

Let a, b, c be real numbers and consider the matrix $M = \begin{bmatrix} 0 & a & 0 \\ b & c & a \\ 0 & b & 0 \end{bmatrix}$.

(a) Determine all values of a, b, c for which M has distinct eigenvalues.

(b) Determine all values of a, b, c for which M is unitarily diagonalizable.

(c) Take $a = 1$, $b = 0$, $c = 1$ and determine the Jordan canonical form of M.

Required Knowledge: **Eigenvalues, eigenvectors, unitary diagonalization, Jordan canonical form.**

Solution:

2a: The charateristic polynomial of M is given by

$$p_M(\lambda) = \begin{vmatrix} \lambda & -a & 0 \\ -b & \lambda - c & -a \\ 0 & -b & \lambda \end{vmatrix} = \lambda^2(\lambda - c) - 2ab\lambda = \lambda(\lambda^2 - c\lambda - 2ab).$$

Therefore, the eigenvalues are $\lambda_1 = 0$ and $\lambda_{2,3} = \frac{c \mp \sqrt{c^2 + 8ab}}{2}$. We can distinguish three cases: $c^2 + 8ab < 0$, $c^2 + 8ab = 0$, and $c^2 + 8ab > 0$.

- Case 1: $c^2 + 8ab < 0$
  In this case, $\lambda_{2,3}$ are nonreal numbers that are conjugate of each other. Therefore, eigenvalues are distinct.

- Case 2: $c^2 + 8ab = 0$
  In this case, $\lambda_2 = \lambda_3$ and hence eigenvalues are not distinct.

- Case 3: $c^2 + 8ab > 0$
  In this case, $\lambda_2 \neq \lambda_3$. Note that $\lambda_2 = 0$ or $\lambda_3 = 0$ if and only if $ab = 0$. Therefore, eigenvalues are distinct if and only if $ab \neq 0$.

Consequently, we see that the eigenvalues of M are distinct if and only if

$$c^2 + 8ab < 0 \text{ or } ab \neq 0.$$

2b: The matrix M is unitarily diagonalizable if and only if $M^T M = M M^T$. Note that

$$M^T M = \begin{bmatrix} 0 & b & 0 \\ a & c & b \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 0 & a & 0 \\ b & c & a \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} b^2 & bc & ab \\ bc & a^2 + b^2 + c^2 & ac \\ ab & ac & a^2 \end{bmatrix}$$

and

$$M M^T = \begin{bmatrix} 0 & a & 0 \\ b & c & a \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} 0 & b & 0 \\ a & c & b \\ 0 & a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & ac & ab \\ ac & a^2 + b^2 + c^2 & bc \\ ab & bc & b^2 \end{bmatrix}.$$

Therefore, we see that $M^T M = M M^T$ if and only if $a^2 = b^2$ and $ac = bc$. Note that $a^2 = b^2$ if and only if $a = b$ or $a = -b$. Also, note that $ac = bc$ if and only if $a = b$ or $c = 0$. Hence, we see that M is unitarily diagonalizable if and only if $a = b$ or $\big((a = -b)$ and $c = 0\big)$.

2c: If $a = 1$, $b = 0$, and $c = 1$, then we have

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this is an upper triangular matrix, we see that the distinct eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$. In order to find out the Jordan canonical form, we need to compute the Weyr characteristics for each distinct eigenvalue.

For $\lambda_1 = 1$, we have

$$\lambda_1 I - M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\operatorname{rank} \lambda_1 I - M = 2.$$

Also, note that

$$(\lambda_1 I - M)^2 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\operatorname{rank}(\lambda_1 I - M)^2 = 2.$$

Therefore, for this eigenvalue we have $w_1 = 3 - 2 = 1$ and $w_2 = 2 - 2 = 0$. This leads to $\kappa = 1$ and $\rho_1 = 1 - 0 = 1$. As expected, the Jordan block for this eigenvalue is 1×1.

For $\lambda_2 = 0$, we have

$$\lambda_2 I - M = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that

$$\operatorname{rank} \lambda_2 I - M = 2.$$

Also, note that

$$(\lambda_2 I - M)^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\operatorname{rank}(\lambda_2 I - M)^2 = 1.$$

Further, we have

$$(\lambda_2 I - M)^3 = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\operatorname{rank}(\lambda_2 I - M)^3 = 1.$$

Therefore, for this eigenvalue we have $w_1 = 3 - 2 = 1$, $w_2 = 2 - 1 = 1$, and $w_3 = 1 - 1 = 0$. This leads to $\kappa = 2$ and hence $\rho_1 = 1 - 1 = 0$ and $\rho_2 = 1 - 0 = 1$. Thus, we see that there is only one 2×2 Jordan block for this eigenvalue.

Consequently, the Jordan form (up to permutations of Jordan blocks) is

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

3 Unitary diagonalization 20 pts

Find an orthogonal diagonalizer for the matrix $M = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Required Knowledge: **Unitary diagonalization.**

Solution:

3: Since M is real symmetric, it is diagonalizable with an orthogonal matrix. To find such a diagonalizer, we proceed with finding the eigenvalues and eigenvectors. The characteristic polynomial is

$$\begin{aligned} p_M(\lambda) = \begin{vmatrix} \lambda-2 & 1 & -1 \\ 1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{vmatrix} &= (\lambda-2)^3 + 1 + 1 - (\lambda-2) - (\lambda-2) - (\lambda-2) \\ &= (\lambda-2)(\lambda^2 - 4\lambda + 4 - 3) + 2 \\ &= (\lambda-2)(\lambda^2 - 4\lambda + 1) + 2 \\ &= \lambda^3 - 4\lambda^2 + \lambda - 2\lambda^2 + 8\lambda - 2 + 2 \\ &= \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2. \end{aligned}$$

Therefore, the distinct eigenvalues are 0 and 3.

For the eigenvalue 0, the eigenvectors satisfy

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}.$$

Note that

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \xrightarrow{①\leftrightarrow②} \begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ -1 & -1 & -2 \end{bmatrix} \xrightarrow[③\leftarrow ③+1\cdot ①]{②\leftarrow ②+2\cdot ①} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{③\leftarrow ③-1\cdot ②} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \xrightarrow{③\leftarrow -\frac{1}{3}\cdot ③} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{①\leftarrow ①+2\cdot ②} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

is an eigenvector for eigenvalue 0. After normalizing, we obtain

$$\boldsymbol{y} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

For the eigenvalue 3, the eigenvectors satisfy

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}.$$

Note that

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[\textcircled{3} \leftarrow \textcircled{3} + 1 \cdot \textcircled{1}]{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{1}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As such,

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors for the eigenvalue 3. To orthonormalize these vectors, we first normalize $\boldsymbol{x}_1$:

$$\boldsymbol{y}_1 = \frac{1}{\|\boldsymbol{x}_1\|} \boldsymbol{x}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Then, we find

$$\boldsymbol{z}_2 = \boldsymbol{x}_2 - (\boldsymbol{y}_1^T \boldsymbol{x}_2) \boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Finally, we have

$$\boldsymbol{y}_2 = \frac{1}{\|\boldsymbol{z}_2\|} \boldsymbol{z}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Therefore, an orthogonal diagonalizer is

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}. \end{bmatrix}$$

Indeed, we have

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}. \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & \frac{6}{\sqrt{6}} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}. \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & \frac{6}{\sqrt{6}} \end{bmatrix}$$

This means that

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}. \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}. \end{bmatrix}^{-1}.$$

4 Characteristic polynomial $\qquad 3 + 2 + 3 + 2 + 10 + 10 = 30$ pts

Let M be an 3×3 matrix with $p_M(\lambda) = (\lambda^2 + \lambda + 1)(\lambda - 1)$.

(a) Find the eigenvalues of M.

(b) Find the trace and determinant of M.

(c) Can M be a Hermitian matrix? Justify your answer.

(d) Is M diagonalizable? Justify your answer.

(e) Find scalars a, b, c such that $M^{11112022} = aM^2 + bM + cI$.

(f) Show that M is similar to M^T.

REQUIRED KNOWLEDGE: **Charasteristic polynomial, eigenvalues, similarity, Cayley-Hamilton theorem.**

SOLUTION:

4a: Since eigenvalues are the roots of the characteristic polynomial, we se that the eigenvalues of M are 1 and $-\frac{1}{2}(1 \mp i\sqrt{3})$.

4b: Since the trace is the sum and the determinant is the product of eigenvalues, we see that they are equal to, respectively, 0 and -1.

4c: Since eigenvalues of Hermitian matrices are real, we see that M cannot be a Hermitian matrix.

4d: Since M has distinct eigenvalues, it is diagonalizable.

4e: Note that $p_M(\lambda) = (\lambda^2 + \lambda + 1)(\lambda - 1) = \lambda^3 - 1$. Therefore, the Cayley-Hamilton theorem implies that $M^3 = I$. Since $11112022 = 3 \cdot 3704007 + 1$, we have $M^{11112022} = M(M^3)^{3704007} = M$. Therefore, we see that $a = c = 0$ and $b = 1$.

4f: Since M is diagonalizable, there exist a nonsingular matrix X and a diagonal matrix D such that $M = XDX^{-1}$. This implies that

$$D = X^{-1}MX \quad \text{and} \quad M^T = (X^{-1})^T D^T X^T. \tag{1}$$

Since D is diagonal, $D = D^T$. Therefore, we have $M^T = (X^{-1})^T X^{-1} M X X^T$. Note that $(XX^T)^{-1} = (X^{-1})^T X$. Consequently, M is similar to M^T.